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ALGEBRAIC SOLUTIONS OF ONE-DIMENSIONAL FOLIATIONS

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Abstract

In this article we consider the problem of extending the result of J.P.Jouanolou on the density of singular holomorphic foliations on $\mathbf{C}P(2)$ without algebraic solutions to the case of foliations by curves of $\mathbf{C}P(n)$.

1. Introduction and statement of results

A one-dimensional (singular) holomorphic foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ is given by a morphism

$$\Upsilon : \mathcal{O}(-d) \longrightarrow T\mathbf{C}P(n)$$

with singular set $sing(\mathcal{F}) = \{p : \Upsilon(p) = 0\}$. We will consider foliations with singular set in codimension greater than 1. Such a foliation $\mathcal{F}$ is represented in affine coordinates $(x_1, \ldots, x_n)$ by a vector field of the form

$$X = gR + \sum_{\ell=0}^{d} X_\ell$$

where R is the radial vector field $R = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$, g is a homogeneous polynomial of degree d and X_ℓ is a vector field whose components are homogeneous polynomials of degree ℓ, $0 \leq \ell \leq d$. Since $sing(\mathcal{F})$ has codimension greater than 1 we have either $g \not\equiv 0$ or $g \equiv 0$ and X_d cannot be written as hR where h is homogeneous of degree $d-1$. In this case X has a pole of order $d-1$ at infinity (see [6]). We call d the *degree* of the foliation.

If $\mathcal{F}$ is a holomorphic foliation of dimension 1 on $\mathbf{C}P(n)$ with singular set $sing(\mathcal{F})$ and $\Gamma \subset \mathbf{C}P(n)$ is an irreducible algebraic curve, we say that Γ is an *algebraic solution* of $\mathcal{F}$ provided $\Gamma \setminus sing(\mathcal{F})$ is a leaf of the foliation. We prove the following :

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Theorem I. *Consider the vector fields*

$$X_0^d = \sum_{i=1}^{n-1}(x_{i+1}^d - x_i x_1^d)\frac{\partial}{\partial x_i} + (1 - x_n x_1^d)\frac{\partial}{\partial x_n}$$

and

$$X_\mu^d = \mu R + X_0^d \qquad \mu \in \mathbf{C}$$

and let $\mathcal{F}_0^d$, $\mathcal{F}_\mu^d$ be the foliations on $\mathbf{C}P(n)$, $n \geq 2$, represented by X_0^d and X_μ^d respectively. Then, for $d \geq 2$ and n even, $\mathcal{F}_0^d$ has no algebraic solution and, for $d \geq 2$ and n odd, $\mathcal{F}_\mu^d$ has no algebraic solution provided $0 <\mid \mu \mid << 1$.

Theorem II. *Let $\aleph_d$ denote the space of one-dimensional holomorphic foliations of degree d on $\mathbf{C}P(n)$, $n \geq 2$. For each $d \geq 2$, there is an open and dense subset $\Im_d \subset \aleph_d$ such that if $\mathcal{F} \in \Im_d$, then $\mathcal{F}$ has no algebraic solution.*

In fact we show that $\mathcal{F}_0^d$ has no algebraic solution of geometric genus greater than 0, whether n is even or odd. The necessity to consider the one-parameter family of vector fields X_μ^d arises from the fact that, when n is odd, $\mathcal{F}_0^d$ has precisely $\frac{d^n+d^{n-1}+\cdots+d+1}{d+1}$ invariant projective lines. Also, it's shown that $\mathcal{F}_\mu^d$ has no algebraic solution, for any $n \geq 2$ and $d \geq 2$, provided $0 <\mid \mu \mid << 1$.

To obtain the results we proceed as follows. The set $\Im_d$ consists of foliations of "generic type" with simple linear singularities at isolated points and Theorem I shows that this set is not empty (the vector field given in this theorem is just a n-dimensional version of the example given by Jouanolou in [8]). That's actually the most involved part of the article. By a foliation $\mathcal{F}$ of "generic type" and degree $d \geq 2$ we mean a foliation represented by a vector field as above and such that (**i**) at each $p \in sing(X)$ we have $det\mathbf{D}X(p) \neq 0$, (**ii**) if $\{\lambda_1^p, \ldots, \lambda_n^p\}$ are the eigenvalues of $\mathbf{D}X(p)$ then they satisfy $\frac{\lambda_i^p}{\lambda_j^p}$ is not a positive real number for $i \neq j$ (**iii**) a finite number of sums of "residues" (which are rational functions of the λ_i^p's), associated to the foliation at singular points, are not certain positive integers and (**iv**) no $d+1$ points in $sing(X)$ lie on a projective line. These are sufficient conditions for the foliation to have no algebraic solutions. A brief explanation of this fact is the following: first recall that if a smooth algebraic curve is invariant by a foliation on $\mathbf{C}P(n)$ then the curve must contain a singular point of the foliation, for otherwise we get a holomorphic foliation with a compact leaf, which is impossible. Now suppose we have an invariant algebraic curve; then (**ii**) says that this curve cannot have singular analytic nor smooth tangent branches at each of its singularities and also that the number of branches at a singular point is bounded by n (Proposition 2.5), so we are reduced to consideration of invariant algebraic curves whose singularities, if any, have only smooth analytic branches no two of which are tangent. In this case we bring in the Theorem of Baum and Bott [1] and a similar result due to D.Lehmann [9]. The idea is that certain characteristic classes of bundles associated to the ambient complex manifold and to the foliation, as well as to invariant submanifolds,"localize" near the singular set of the foliation, giving rise to residues computable through local data for the foliation and whose sum give characteristic numbers of these bundles. Condition **(iii)** means precisely that the sum of residues cannot be a characteristic number associated to a convenient bundle, thus rulling out the existence of certain algebraic solutions. Condition **(iv)** is typical of the odd dimensional situation and aimed at avoiding the existence of any invariant linearly embedded $\mathbf{C}P(1)$.

In [8] Jouanolou proved both theorems for $\mathbf{C}P(2)$ (except that $\Im_d$ is open). Later the first author reproved both theorems (adding the fact that $\Im_d$ is open) [10], and the arguments of the proofs were based on residues associated to foliations. More recently the second author extended both results to foliations on $\mathbf{C}P(3)$ and also showed that any $\mathcal{F} \in \Im_d$ has no invariant algebraic surface, although in this case $\Im_d$ is just proven to be dense [11].

2. Auxiliary results

We start by recalling the theorem of Baum-Bott as written by Chern in [2]. Let $\mathcal{W}$ be a compact complex manifold of dimension n and $\mathcal{L}$ be a holomorphic line bundle on $\mathcal{W}$.

A holomorphic section $X \in \Gamma(T\mathcal{W} \otimes \mathcal{L})$ is given locally by

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

where $x_1, \ldots, x_n$ form a local coordinate system in $\mathcal{W}$ and $X_i \in \Gamma(\mathcal{L})$ are holomorphic sections of $\mathcal{L}$. Suppose the vector field X has only non-degenerated singularities, that is, if $X(p) = 0$ then the matrix $\mathcal{J}_p = (\frac{\partial X_i}{\partial x_j}(p))$ is such that $det\mathcal{J}_p \neq 0$. Consider the Chern classes of the virtual bundle $T\mathcal{W} - \mathcal{L}^{-1}$

$$c_k(T\mathcal{W} - \mathcal{L}^{-1}) = c_k(\mathcal{W}) + c_{k-1}(\mathcal{W})c_1(\mathcal{L}) + \cdots + (c_1(\mathcal{L}))^k, 1 \leq k \leq n$$

and let

$$c^{\alpha}(T\mathcal{W} - \mathcal{L}^{-1}) = c_1^{\alpha_1}(T\mathcal{W} - \mathcal{L}^{-1}) \dots c_n^{\alpha_n}(T\mathcal{W} - \mathcal{L}^{-1})$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n) \qquad \alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n$$

Then we have the

Theorem 2.1 [2].

$$\int_{\mathcal{W}} c^{\alpha}(T\mathcal{W} - \mathcal{L}^{-1}) = \sum \frac{c^{\alpha}(\mathcal{J}_p)}{det\mathcal{J}_p}$$

where the summation extends over all singularities of X.

Let $\mathcal{W}$ be a n-dimensional complex manifold, $\mathcal{F}$ a one-dimensional singular holomorphic foliation on $\mathcal{W}$ with $sing(\mathcal{F})$ a discrete set of points and $V \subset \mathcal{W}$ a complex submanifold invariant by $\mathcal{F}$ with $dim_{\mathbf{C}} V = m$. For each point $p \in sing(\mathcal{F})$ take a coordinate domain $\mathcal{U}$ around p with $\mathcal{U} \cap sing(\mathcal{F}) = \{p\}$ and such that $U = V \cap \mathcal{U}$ is given by $y_1 = \dots = y_q = 0$ where $(x_1, \dots, x_m, y_1, \dots, y_q)$ are coordinates in $\mathcal{U}$, $p = (0, \dots, 0)$ in these coordinates and $m + q = n$. Let the foliation $\mathcal{F}$ be represented in $\mathcal{U}$ by the vector field

$$X = \sum_{i=1}^{m} A_i(x,y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{q} B_j(x,y)\frac{\partial}{\partial y_j}$$

where $B_j(x,0) = 0$ for $1 \leq j \leq q$. If $\varphi \in \mathbf{R}[c_1, \dots, c_q]$ is a characteristic class of dimension $2m$, $\mathcal{J}(x)$ is the matrix $(\frac{\partial B_i}{\partial y_j}(x,0))$,$1 \leq i,j \leq q$, and if we define

$$Res_{\mathcal{F}}(\varphi, V, p) = \begin{bmatrix} \varphi(\mathcal{J}(x))dx_1 \wedge \dots \wedge dx_m \\ A_1(x,0), \dots, A_m(x,0) \end{bmatrix}_0$$

where $[\dots]_0$ denotes the Grothendieck residue symbol at 0 then we have, provided V is compact, the following

Theorem 2.2 [9].

$$\int_V \varphi(\nu_{V/\mathcal{W}}) = \sum_{p \in sing(\mathcal{F}) \cap V} Res_{\mathcal{F}}(\varphi, V, p)$$

where the integral is over the fundamental class of V and $\nu_{V/\mathcal{W}}$ is the normal bundle of V in $\mathcal{W}$.

Remark 2.3 If a vector field X has non-degenerated linear part at a singular point p, $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the linear part of X

at this point and if V is one-dimensional, invariant by X and tangent at p to the direction associated to λ_i then, by taking $\varphi = c_1$ we have

$$Res_X(c_1, V, p) = \begin{bmatrix} c_1(\mathcal{J}(x))dx_i \\ A_i(x,0) \end{bmatrix} = \frac{\sum_{i\neq j}\lambda_j}{\lambda_i}$$

(*see* [1] *or* [5, p. 658]).

We will also need the following Propositions (Propositions 2.4 and 2.7 appeared in [11] and a two-dimensional version of Proposition 2.8 appeared in [3], but the proof we give here is more general):

Proposition 2.4 [11]. *Let $\Gamma \subset \mathbf{C}P(n)$ be an irreducible algebraic curve whose singularities, in case they exist, are such that Γ has only smooth analytic branches, no two of which are tangent, through each of them. Suppose $sing(\Gamma) \subset \{p_1, \ldots, p_m\}$ and consider the sequence of blow-ups*

$$\mathbf{C}P(n) := M_0 \overset{\pi_1}{\longleftarrow} M_1 \overset{\pi_2}{\longleftarrow} M_2 \ldots \overset{\pi_m}{\longleftarrow} M_m := \mathcal{M}$$

where M_i is obtained by blowing-up M_{i-1} at $\pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}(p_i)$. Let $\Gamma^\star \subset \mathcal{M}$ be the proper transform of Γ. Then

$$\int_{\Gamma^\star} c_1(\nu_{\Gamma^\star/\mathcal{M}}) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star) - (n-1)\sum_{i=1}^{m} \ell(p_i)$$

where $d^0(\Gamma)$ is the degree of Γ, $\chi(\Gamma^\star) = 2-2g$ is the Euler characteristic of $\Gamma^\star$ and $\ell(p_i)$ is the number of analytic branches of Γ through p_i.

Let X be a holomorphic vector field defined in a neighborhood U of $0 \in \mathbf{C}^n$ and such that $X(0) = 0$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of $\mathbf{D}X(0)$. An invariant branch for X at 0 is, by definition, a germ of a irreducible non-constant curve Γ through $0 \in U$ such that for each $p \in \Gamma \setminus \{0\}$ we have $X(p) \in T_p\Gamma$. Then we have the following

Proposition 2.5. *Let X, $\lambda_1, \ldots, \lambda_n$ be as above. Suppose $\lambda_1, \ldots, \lambda_n \neq 0$ and that, for $i \neq j$, $\frac{\lambda_i}{\lambda_j} \notin \mathbf{R}^+$. Then X has exactly n invariant branches through 0, say $B_1, \ldots, B_n$ such that:*

(i) *$B_1, \ldots, B_n$ are smooth at 0.*

(*ii*) *For each eigendirection of $\mathbf{D}X(0)$, say e_j, there is exactly one $i \in \{1, \ldots, n\}$, such that B_i is tangent to e_j at 0.*

(*iii*) *If B is a invariant branch for X at 0, then $B = B_j$ for some j (as germs at 0).*

Proof. Set $S = \{\lambda_1, \ldots, \lambda_n\}$. The hypothesis imply that $\mathbf{D}X(0)$ is diagonalizable and that for each λ_j there is a unique eigendirection $\mathbf{C}.e_j$, where $e_j \in \mathbf{C}^n \setminus \{0\}$. If $I \subset \{1, \ldots, n\}$, $I \neq \emptyset$, we will use the notation E_I for the subspace generated by the set $\{e_i \mid i \in I\}$. If $I = \emptyset$ we set $E_I = \{0\}$. We need a lemma:

Lemma. *Let ℓ be a straight line through $0 \in \mathbf{C}$ such that $\ell \cap S = \emptyset$ and the components of $\mathbf{C} - \ell$ are A_1 and A_2. Set $I_k = \{j \mid \lambda_j \in A_k\}$ for $k = 1, 2$. Then there are germs of holomorphic submanifolds $W_k, k = 1, 2$, through $0 \in \mathbf{C}^n$, such that:*

(1) $T_0(W_k) = E_{I_k}, k = 1, 2$

(2) W_k is invariant for X,

(3) If $X^k = X_{|W_k}$ then $0 \in W_k$ is a singularity of X^k of Poincare type, $k = 1, 2$. Moreover the spectrum of $\mathbf{D}X^k(0)$ is $S_k = \{\lambda_j \mid j \in I_k\}$.

(4) If B is a invariant branch for X at 0, then either $B \subset W_1$ or $B \subset W_2$.

Proof. Let $\alpha \in \mathbf{C}^*$ be such that $\alpha.\ell$ is the imaginary axis and $Y = \alpha.X$. Then the imaginary axis divides the spectrum of Y into two parts, namely $S_s = \{\alpha\lambda_j \mid Re(\alpha\lambda_j) < 0\}$ and $S_u = \{\alpha\lambda_j \mid Re(\alpha\lambda_j) > 0\}$. One of these parts, say S_s, corresponds to I_1 and the other to I_2. Let Y_t be the local real flow generated by Y. It is not difficult to see that for each $t > 0$, the local diffeomorphism $Y_t : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}^n, 0)$ is holomorphic and has a hyperbolic fixed point at $0 \in \mathbf{C}^n$. Moreover, the stable subspace of $\mathbf{D}Y_t(0)$ is E_{I_1} and the unstable is E_{I_2}. Let W_1 and W_2 be the stable and unstable manifolds of Y_t, respectively. Then $T_0(W_k) = E_{I_k}$, for $k = 1, 2$. On the other hand, since Y_t is a holomorphic local diffeomorphism, the proof of the existence of the local unstable manifold of [7] (by the graph transformation), implies that W_1 and W_2 are in fact holomorphic submanifolds. Let us prove that W_1 and W_2 are invariant for X. Let ℓ' be another straight line through $0 \in \mathbf{C}$, such that the components of $\mathbf{C} - \ell'$, say $A_1{}'$ and $A_2{}'$, satisfy $A_k{}' \cap S = A_k \cap S$, $k = 1, 2$ (for instance, a small perturbation of ℓ). Let $\beta \in \mathbf{C}^*$ be such that $\beta\ell'$ is the imaginary axis and $\beta A_1{}' = \{z \mid Re(z) < 0\}$. If $Z = \beta X$ and Z_s is its local real flow then we have:

(i) Z_s and Y_t commute (because $Y_t = X_{\alpha t}$ and $Z_s = X_{\beta s}$, where X_T is the local complex flow of X).

(ii) If $W_1{}'$ and $W_2{}'$ are the stable and unstable manifolds of Z_s , $(s > 0)$ respectively, then $T_0(W_k{}') = E_{I_k} = T_0(W_k)$, $k = 1, 2$.

Now, (i) and (ii) imply that $W_k{}' = W_k$, $k = 1, 2$. Moreover, since Y_t and Z_s generate the orbits of the complex flow X_T (considered as a local $\mathbf{R}^2$ action), it follows that the orbits of X_T through points of W_k are contained in W_k, $k = 1, 2$. This implies assertion (2) of the lemma. Assertion (3) follows from the fact that the spectrum of $\mathbf{D}X(0)_{|E_{I_k}}$ is $A_k \cap S$, $k = 1, 2$. It remains to prove assertion (4). Let B be a invariant branch for X at 0. Let $\gamma : (\mathbf{C}, 0) \longrightarrow (\mathbf{C}^n, 0)$ be a Puiseux's parametrization of B. Then it is possible to define a holomorphic vector field X^* on a neighborhood of $0 \in \mathbf{C}$, such that $\mathbf{D}\gamma.X^* = X \circ \gamma$. This vector field X^* is of the form $X^* = z^k u(z)\frac{\partial}{\partial z}$, where $u(0) \neq 0$ and $k \geq 1$ (see [4]). Let $\alpha \in \mathbf{C}^*$, $Y = \alpha X$ and Y_t be as before. If $Y^* = \alpha X^*$ and Y_t^* is the local real flow associated to Y^*, then we must have

$$Y_t^*(z) = \gamma^{-1}(Y_t(\gamma(z)))$$

Let us suppose, by contradiction, that $B \not\subset W_1 \cup W_2$. Since W_1 and W_2 are the stable and unstable manifolds, respectively, of Y_t, it follows that if δ is an orbit of Y_t^* through a point $z_0 \neq 0$, then:

(a) δ is not a closed orbit.

(b) δ cannot accumulate at 0.

On the other hand δ is a solution of the real differential equation

$$\frac{\mathbf{d}z}{\mathbf{d}t} = \alpha z^k u(z)$$

and it's not difficult to see that:

(c) If $k \geq 2$ or $k = 1$ and $Re(\alpha u(0)) \neq 0$ then δ accumulates at 0.

(d) If $k = 1$ and $Re(\alpha u(0)) = 0$ then δ is closed (if z_0 is near 0).

This contradiction implies that $B \subset W_1 \cup W_2$. Since B is irreducible we must have either $B \subset W_1$ or $B \subset W_2$. This proves the Lemma.

Let us prove the existence of the branches $B_1, \ldots, B_n$. Let $\lambda_j = \rho_j e^{\sqrt{-1}\theta_j}$, $0 \leq \theta_j < 2\pi$. Hypothesis (b) implies that $\theta_i \neq \theta_j$ for $i \neq j$, so we may assume, without loss of generality, that $\theta_j < \theta_{j+1}$, for $1 \leq j \leq n-1$. For the existence of B_j, we take a straight line ℓ through $0 \in \mathbf{C}$ such that $\ell \cap S = \emptyset$ and λ_j , λ_{j+1} belong to different components of $\mathbf{C} - \ell$, say $\lambda_j \in A_1$ and $\lambda_{j+1} \in A_2$, where $\mathbf{C} - \ell = A_1 \cup A_2$ (if $j = n$ we take $\lambda_{j+1} = \lambda_1$). In this case, if I_1 and I_2 are as in the Lemma, then $j \in I_1$ and $j+1 \in I_2$. Let W_1 and X^1 be as in the Lemma. Observe that W_1 is biholomorphically equivalent to an open set in $\mathbf{C}^k$, $k < n$, 0 is a singularity of X^1 of Poincare type, and $\lambda_j \in S_1$, the spectrum of $\mathbf{D}X^1(0)$. Moreover, it follows from the construction of ℓ that it is possible to find a straight line ℓ' through $0 \in \mathbf{C}$ such that $\ell' \cap S_1 = \emptyset$, and if A_1' and A_2' are the connected components of $\mathbf{C} - \ell'$, then $S_1 \cap A_1' = \{\lambda_j\}$. Applying once again the Lemma for X^1 we get the existence of B_j, of dimension 1 and tangent to the eigendirection of λ_j. Now let B be any invariant branch for X at 0. Fix a straight line ℓ as in the Lemma, in such a way that $A_1 \cap S \neq \emptyset$ and $A_2 \cap S \neq \emptyset$, so that $dim W_1 < n$ and $dim W_2 < n$. It follows from the Lemma that either $B \subset W_1$ or $B \subset W_2$. Suppose for instance that $B \subset W_1$. If $dim W_1 = 1$, it's clear that B is an open set in W_1, so we can suppose that in fact $B = W_1$. If $dim W_1 > 1$ we can apply the same argument to show that $B \subset W_1^1$, where W_1^1 is invariant for X, smooth , and $dim W_1^1 < dim W_1$. It is clear that after repeating this argument a finite number of times we will get $B \subset W_1^k$, where W_1^k is invariant for X and $dim W_1^k = 1$. On the other hand, it follows from the construction of $B_1, \ldots, B_n$ and from the above argument that $W_1^k = B_j$ for some j. Therefore $B \subset B_j$ for some j, which implies that, as germs of curves, we must have $B = B_j$.

In particular we have the

Corollary 2.6. *If X and $\lambda_1, \ldots, \lambda_n$ are as in Proposition 2.5, then X has no invariant singular branch at $0 \in \mathbf{C}^n$.*

Let us consider now a one-dimensional singular holomorphic foliation $\mathcal{F}$ on $\mathbf{C}P(n)$ with $sing(\mathcal{F})$ a finite set of points and such that if X_p is a vector field representing $\mathcal{F}$ in a neighborhood of $p \in sing(\mathcal{F})$ then p is a non-degenerated singularity of X_p and further, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{D}X_p(p)$ satisfy

$$\frac{\lambda_i}{\lambda_j} \notin \mathbf{R}^+, i \neq j$$

Let $\Gamma \subset \mathbf{C}P(n)$ be as is Proposition 2.4 and suppose Γ is invariant by $\mathcal{F}$. For each $p \in sing(\mathcal{F}) \cap \Gamma$ let $\mathcal{B}_p$ denote the set of analytic branches of Γ through p and note that since Γ is invariant by $\mathcal{F}$, if $p \in sing(\Gamma)$ then $p \in sing(\mathcal{F})$. We have the

Proposition 2.7 [11].*The following equality holds :*

$$\sum_{p \in sing(\mathcal{F}) \cap \Gamma} \sum_{B \in \mathcal{B}_p} Res_{\mathcal{F}}(c_1, B, p) = (n+1)d^0(\Gamma) - \chi(\Gamma^*)$$

Let $\Gamma \subset \mathbf{C}P(n)$ and $\mathcal{F}$ be as in Propositions 2.4 and 2.7, let $\mathcal{F}$ have degree d and for each $p \in sing(\mathcal{F}) \cap \Gamma$ let $\ell(p)$ be the number of analytic branches of Γ through p. Then

Proposition 2.8.

$$\chi(\Gamma^*) = \left(\sum_{p \in sing(\mathcal{F}) \cap \Gamma} \ell(p) \right) - (d-1)d^0(\Gamma)$$

Proof. Choose a hyperplane H_∞ such that $sing(\mathcal{F}) \cap \Gamma \subset \mathbf{C}^n = \mathbf{C}P(n) \setminus H_\infty$, H_∞ intersects Γ transversely and the foliation $\mathcal{F}$ is represented in $\mathbf{C}^n$ by a vector field X which has a pole of order $d-1$ at H_∞. Blow-up $\mathbf{C}P(n)$ at each point $p \in sing(\mathcal{F}) \cap \Gamma$ and obtain a manifold $\mathcal{M}$ as in Proposition 2.4. Let X^* be a lifting of X to $\mathcal{M}$. Then X^* induces a meromorphic vector field on the strict transform Γ^* of Γ, say $\mathcal{V}$, with the following properties:

(i) $\mathcal{V}$ has $\sum_{p \in sing(\mathcal{F}) \cap \Gamma} \ell(p)$ zeros.

(ii) $\mathcal{V}$ has $d^0(\Gamma)$ poles of order $d-1$, corresponding to the $d^0(\Gamma)$ intersections of H_∞ with Γ.

Let $\mathcal{L}$ denote the pull-back to $\Gamma^\star$ of the bundle $[H_\infty]$ by the blow-up mapping
$\pi : \mathcal{M} \longrightarrow \mathbf{C}P(n)$. Applying Theorem 2.1 to the section $\mathcal{V} \in \Gamma(T\Gamma^\star \otimes \mathcal{L}^{\otimes(d-1)})$ we have

$$\int_{\Gamma^\star} c_1(T\Gamma^\star - (\mathcal{L}^{\otimes(d-1)})^{-1}) = \int_{\Gamma^\star} c_1(\Gamma^\star) + (d-1)\int_{\Gamma^\star} c_1(\mathcal{L})$$
$$= \chi(\Gamma^\star) + (d-1)d^0(\Gamma)$$

and

$$\int_{\Gamma^\star} c_1(T\Gamma^\star - (\mathcal{L}^{\otimes(d-1)})^{-1}) = \sum_{q \in sing(\mathcal{V})} \frac{det\mathcal{J}_q}{det\mathcal{J}_q} = \sum_{p \in sing(\mathcal{F})\cap\Gamma} \ell(p)$$

and the result follows.

3. Proof of Theorem I

Recall the vector field

$$X_0^d = \sum_{i=1}^{n-1}(x_{i+1}^d - x_i x_1^d)\frac{\partial}{\partial x_i} + (1 - x_n x_1^d)\frac{\partial}{\partial x_n} \quad , \quad d \geq 2$$

The foliation $\mathcal{F}_0^d$ defined by X_0^d in $\mathbf{C}P(n)$ has no singularities at infinity, as is easily verified and the singular set $sing(\mathcal{F}_0^d)$ consists of

$$D = d^n + d^{n-1} + \cdots + d + 1$$

points $p_i = (x_{1,i}, \ldots, x_{n,i})$, $1 \leq i \leq D$. In fact, the singularities are given by the roots of

$$x_1^{d^n + d^{n-1} + \cdots + d + 1} = 1$$

with

$$x_{n-j} = x_1^{-d^{j+1} - d^j \cdots - d}, \quad 0 \leq j \leq n-2$$

So that, if ξ is a primitive root of unity of order D then

$$sing(\mathcal{F}_0^d) = \{p_i = (\xi^i, \xi^{-i(d^{n-1}+\cdots+d)}, \ldots, \xi^{-i(d^2+d)}, \xi^{-id}) : \quad 1 \leq i \leq D\}$$

Remark 3.1. Let $[\alpha_1, \ldots, \alpha_{n+1}] \in PGL(n+1, \mathbf{C})$ denote the class of the matrix $diag(\alpha_1, \ldots, \alpha_{n+1})$ and $\mathcal{H} \subset PGL(n+1, \mathbf{C})$ be the subgroup consisting of the elements $[\alpha_1, \ldots, \alpha_{n+1}]$ which satisfy

$$\frac{\alpha_i}{\alpha_{i+1}} = \frac{\alpha_{i+1}^d}{\alpha_{i+2}^d}, \quad 1 \leq i \leq n-1, \quad \frac{\alpha_n}{\alpha_{n+1}} = \frac{\alpha_{n+1}^d}{\alpha_1^d}, \quad d \geq 2$$

Then, it's easy to see that the group $\mathcal{H}$ is cyclic of order D and generated by the class of

$$[\xi, \xi^{-(d^{n-1}+\cdots+d)}, \ldots, \xi^{-(d^2+d)}, \xi^{-d}, 1]$$

where ξ is a primitive root of unity of order D and moreover, $\mathcal{H}$ acts freely and transitively on $sing(\mathcal{F}_0^d)$.

If $\wp_i(\lambda)$ denotes the characteristic polynomial of $\mathbf{D}X_0^d(p_i)$ where $p_i \in sing(\mathcal{F}_0^d)$, then a calculation shows that

$$\wp_i(\lambda) = (\lambda + x_{1,i}^d)^n + dx_{1,i}^d(\lambda + x_{1,i}^d)^{n-1} + \sum_{j=2}^{n} d^j x_{1,i}^{-D+jd}(\lambda + x_{1,i}^d)_n^{n-j}$$

Since $x_{1,i} = \xi^i$ put $t = \lambda + \xi^{id}$ to get

$$\wp_i(t - \xi^{id}) = t^n + d(t-\lambda)t^{n-1} + d^2(t-\lambda)^2 t^{n-2} + \cdots + d^n(t-\lambda)^n$$

Set $\zeta = \frac{t}{d(t-\lambda)}$ and this polynomial becomes (up to a multiplicative constant):

$$\zeta^n + \zeta^{n-1} + \cdots + \zeta + 1$$

Hence, if ω is a primitive root of unity of order $n+1$, then the spectrum of $\mathcal{F}_0^d$ at p_i is

$$spec(\mathcal{F}_0^d, p_i) = \{\lambda_j^i = (-1 + d\omega^j)\xi^{id} : \quad 1 \le j \le n\}$$

Lemma 3.2. *If $d \ge 2$, then all singularities of $\mathcal{F}_0^d$ satisfy the hypothesis of Proposition 2.5.*

Proof. To see this note that if $p_i \in sing(\mathcal{F}_0^d)$ and $\lambda_j^i, \lambda_k^i \in spec(\mathcal{F}_0^d, p_i)$, $j \ne k$ we have

$$\frac{\lambda_k^i}{\lambda_j^i} = \frac{-1 + d\omega^k}{-1 + d\omega^j}$$

and that the numbers $-1 + d\omega^j$ lie on a circle of radius $d \ge 2$ centered at -1. Hence, if

$$\frac{-1 + d\omega^k}{-1 + d\omega^j} = \alpha \in \mathbf{R}$$

then $\alpha < 0$.

Proposition 3.3. *Suppose $\Gamma \subset \mathbf{C}P(n)$ is an irreducible algebraic curve of genus $g > 0$ whose singularities, in case they exist, are such that Γ has only smooth analytic branches, no two of which are tangent, through each of them. Then Γ cannot be an algebraic solution of $\mathcal{F}_0^d$.*

Proof. If $p \in sing(\Gamma)$ then $p \in sing(\mathcal{F}_0^d) \cap \Gamma$ and note that $sing(\mathcal{F}_0^d) \cap \Gamma \ne \emptyset$ for a holomorphic foliation on $\mathbf{C}P(n)$ has no compact leaf (see [8]). So let

$$sing(\mathcal{F}_0^d) \cap \Gamma = \{q_1, \ldots, q_N\}, \qquad 1 \le N \le D$$

and

$$\mathcal{B}(q_i) = \{B_1^i, \ldots, B_{r(i)}^i\}, \qquad 1 \le i \le N$$

denote the set of analytic branches of Γ through q_i. By Proposition 2.5 and Lemma 3.1 we have that such a branch is necessarily smooth and tangent to the direction associated to an eigenvalue of $\mathrm{D}X_0^d(q_i)$ and $r(i) \le n$. Let us say that B_m^i is tangent to the direction associated to $\lambda_{j_m}^{s_i} = (-1 + d\omega^{j_m})\xi^{s_i d}$. By Remark 2.3 we have

$$Res_{\mathcal{F}_0^d}(c_1, B_m^i, q_i) = \frac{\sum_{k \ne j_m} \lambda_k^{s_i}}{\lambda_{j_m}^{s_i}}$$

and

$$\frac{\sum_{k \ne j_m} \lambda_k^{s_i}}{\lambda_{j_m}^{s_i}} = \sum_{k \ne j_m} \frac{-1 + d\omega^k}{-1 + d\omega^{j_m}}$$

Now

$$\begin{aligned} \sum_{k \ne j_m} (-1 + d\omega^k) = -(n-1) + d\left(\sum_{k \ne j_m} \omega^k\right) &= -(n+d-1) - d\omega^{j_m} \\ &= -(n+d) + 1 - d\omega^{j_m} \end{aligned}$$

so that

$$Res_{\mathcal{F}_0^d}(c_1, B_m^i, q_i) = \sum_{k \ne j_m} \frac{-1 + d\omega^k}{-1 + d\omega^{j_m}} = -1 + \frac{n+d}{1 - d\omega^{j_m}}$$

and

$$Res_{\mathcal{F}_0^d}(c_1, \Gamma, q_i) = \sum_{B_m^i \in \mathcal{B}(q_i)} Res_{\mathcal{F}_0^d}(c_1, B_m^i, q_i)$$

hence

$$(1) \quad Res_{\mathcal{F}_0^d}(c_1, \Gamma, q_i) = \sum_{m=1}^{r(i)} \left(-1 + \frac{n+d}{1 - d\omega^{j_m}}\right) = -r(i) + \sum_{m=1}^{r(i)} \frac{n+d}{1 - d\omega^{j_m}}$$

By Proposition 2.7

$$\sum_{q_i} Res_{\mathcal{F}_0^d}(c_1, \Gamma, q_i) = (n+1)d^0(\Gamma) - \chi(\Gamma^*)$$

where the summation extends over all points $q_i \in sing(\mathcal{F}_0^d) \cap \Gamma$ and by Proposition 2.8

$$(2) \qquad \chi(\Gamma^*) = \left(\sum_{q_i} r(i)\right) - (d-1)d^0(\Gamma)$$

which gives

$$\sum_{q_i} Res_{\mathcal{F}_0^d}(c_1, \Gamma, q_i) = (n+d)d^0(\Gamma) - \sum_{q_i} r(i) \tag{3}$$

Combining (1) and (3) we get

$$-\sum_{q_i} r(i) + \sum_{q_i} \sum_{m=1}^{r(i)} \frac{n+d}{1-d\omega^{j_m}} = (n+d)d^0(\Gamma) - \sum_{q_i} r(i)$$

so that

$$d^0(\Gamma) = \sum_{q_i} \sum_{m=1}^{r(i)} \frac{1}{1-d\omega^{j_m}} \tag{4}$$

and from (2) we deduce

$$d^0(\Gamma) = \frac{\sum_{q_i} r(i) - \chi(\Gamma^\star)}{d-1} = \frac{\sum_{q_i} r(i) - 2 + 2g}{d-1} \tag{5}$$

where g is the genus of $\Gamma^\star$. Now observe that

$$\mid 1 - d\omega^{j_m} \mid > d - 1$$

because $1 \leq j_m \leq n$ and then $\omega^{j_m} \neq 1$. Hence

$$\frac{1}{\mid 1 - d\omega^{j_m} \mid} < \frac{1}{d-1}$$

and from (4) we get

$$d^0(\Gamma) < \frac{\sum_{q_i} r(i)}{d-1} \tag{6}$$

Now, if $g > 0$ (5) and (6) imply

$$d^0(\Gamma) = \frac{\sum_{q_i} r(i) - 2 + 2g}{d-1} \geq \frac{\sum_{q_i} r(i)}{d-1} > d^0(\Gamma) \tag{7}$$

a contradiction which proves the Proposition when $g > 0$.

It remains to consider the case $g = 0$. If ζ is a point in the unit circle then its complex conjugate is ζ^{-1}. From (4) we have

$$\begin{aligned} 2d^0(\Gamma) &= \sum_{q_i} \sum_{m=1}^{r(i)} \left(\frac{1}{1-d\omega^{j_m}} + \frac{1}{1-d\omega^{-j_m}} \right) \\ &= \sum_{q_i} \sum_{m=1}^{r(i)} \left(\frac{2 - d(\omega^{j_m} + \omega^{-j_m})}{1 - d(\omega^{j_m} + \omega^{-j_m}) + d^2} \right) \end{aligned}$$

Write $\omega^{j_m} = c_{j_m} + \sqrt{-1}s_{j_m}$ so that $\omega^{j_m} + \omega^{-j_m} = 2c_{j_m}$ and $-1 \leq c_{j_m} \leq 1$. Then

$$2d^0(\Gamma) = \sum_{q_i} \sum_{m=1}^{r(i)} \left(\frac{2 - 2dc_{j_m}}{1 - 2dc_{j_m} + d^2} \right) \Longrightarrow d^0(\Gamma) = \sum_{q_i} \sum_{m=1}^{r(i)} \frac{1 - dc_{j_m}}{1 - 2dc_{j_m} + d^2}$$

Let us consider the function $f : [-1, 1] \longrightarrow \mathbf{R}$ given by

$$f(t) = \frac{1 - td}{1 - 2td + d^2}, \qquad d \geq 2$$

Its derivative is

$$f'(t) = \frac{-d(d^2 - 1)}{(1 - 2td + d^2)^2} < 0, \qquad \forall t \in [-1, 1]$$

and f attains its maximum at $t = -1$ and $f(-1) = \frac{1}{d+1}$. Hence

$$\frac{1 - dc_{j_m}}{1 - 2dc_{j_m} + d^2} \leq \frac{1}{d+1} \Longrightarrow d^0(\Gamma) = \sum_{q_i} \sum_{m=1}^{r(i)} \frac{1 - dc_{j_m}}{1 - 2dc_{j_m} + d^2} \leq \frac{\sum_{q_i} r(i)}{d+1}$$

and since $\chi(\Gamma^\star) = 2$ it follows from (2) that

$$\text{(8)} \qquad d^0(\Gamma) = \frac{\sum_{q_i} r(i) - 2}{d - 1} \leq \frac{\sum_{q_i} r(i)}{d+1} \Longrightarrow \sum_{q_i} r(i) \leq d + 1$$

and this gives

$$\text{(9)} \qquad d^0(\Gamma) = \frac{\sum_{q_i} r(i) - 2}{d - 1} \leq 1$$

so that we have $\mathbf{C}P(1)$ linearly embedded in $\mathbf{C}P(n)$ and a solution of $\mathcal{F}_0^d$. Recall that by Proposition 2.8 we have

$$\text{(10)} \qquad \sum_{q_i} r(i) = \chi(\mathbf{C}P(1)) + (d-1)d^0(\Gamma) = 2 + d - 1 = d + 1$$

so that $sing(\mathcal{F}_0^d) \cap \mathbf{C}P(1)$ consists of precisely $d+1$ points. In this case we must consider the one-parameter family of vector fields $X_\mu^d = \mu R + X_0^d$, $\mu \in \mathbf{C}$ and the associated family $\mathcal{F}_\mu^d$ of foliations on $\mathbf{C}P(n)$. We have the following

Lemma 3.4. *A $\mathbf{C}P(1)$ linearly embedded in $\mathbf{C}P(n)$ meets $sing(\mathcal{F}_0^d)$ in $d+1$ points if and only if n is odd. Moreover, for $0 <| \mu |<< 1$, $sing(\mathcal{F}_\mu^d)$ does not have $d+1$ points aligned.*

Proof. Suppose we have a $\mathbf{C}P(1)$ linearly embedded and such that

$$\mathbf{C}P(1) \cap sing(\mathcal{F}_0^d) = \{q_1, \ldots, q_{d+1}\}$$

Acting by $\mathcal{H}$ (see Remark 3.1) we may assume $q_1 = (1, \dots, 1)$ and write $q_i = (z_i, \dots, z_i^{-d})$ for $1 \leq i \leq d+1$ where z_i is a root of unity of order D and $z_i \neq z_j$ for $i \neq j$. Then there are complex numbers t_i such that

$$q_1 + t_i(q_2 - q_1) = q_i, \qquad 2 \leq i \leq d+1$$

and this gives, looking at the first and at the last coordinates of these points

$$1 + t_i(z_2 - 1) = z_i, \qquad 1 + t_i(z_2^{-d} - 1) = z_i^{-d}$$

Eliminating t_i and taking conjugates we get

$$(*) \quad z_2(1 + z_2 + \cdots + z_2^{d-1}) = z_i(1 + z_i + \cdots + z_i^{d-1}) \qquad 2 \leq i \leq d+1$$

Consider the polynomial $Q(T) = T^d + \cdots + T - \alpha$ where $\alpha = z_2 + z_2^2 + \cdots + z_2^d$. From $(*)$ we deduce $Q(T) = (T - z_2) \dots (T - z_{d+1})$ and $\alpha = (-1)^{d-1} z_2 \dots z_{d+1}$ so that $\mid \alpha \mid = 1$ since z_i is a root of unity. By $(*)$

$$\frac{\mid z_i^d - 1 \mid}{\mid z_i - 1 \mid} = \mid \alpha \mid = 1$$

which is equivalent to

$$\mid z_i^d - 1 \mid = \mid z_i - 1 \mid, \qquad 2 \leq i \leq d+1$$

Now, for each i this implies that either $z_i^d = z_i$ or $z_i^d = z_i^{-1}$ since z_i is a point on the unit circle. If we had $z_i^{d-1} = 1$ for all i then from $(*)$ we would have $z_2 = \cdots = z_{d+1} = \alpha$, a contradiction. If for $i \neq j$ we had $z_i^{d-1} = 1$ and $z_j^{d+1} = 1$ then from $(*)$ we would have $z_i = -1 = \alpha$, so that $z_i^2 = 1$ and hence $z_i^{d+1} = 1$. It's enough to consider the case $z_i^{d+1} = 1$ for all i. But then $d+1$ divides $D = d^n + d^{n-1} + \cdots + d + 1$ and this happens if and only if n is odd. Note that this argument shows that $sing(\mathcal{F}_0^d)$ cannot have ℓ points aligned if $2 < \ell < d+1$ or if $d+1 < \ell \leq D$.

Now let ϱ be a primitive root of unity of order $d+1$ and let n be an odd integer. Then the points $q_i \in sing(\mathcal{F}_0^d)$ do all lie on a projective line in $\mathbf{C}P(n)$ where

$$q_i = (\varrho^i, 1, \varrho^i, \dots, 1, \varrho^i) \quad 0 \leq i \leq d$$

Now, if ϱ is as above and n is odd then it's easily seen that the projective line parametrized by

$$L(t) = (1 + t(\varrho - 1), 1, 1 + t(\varrho - 1), \dots, 1, 1 + t(\varrho - 1)) \quad t \in \mathbf{C}$$

is invariant by X_0^d and therefore an algebraic solution of $\mathcal{F}_0^d$. As a matter of fact, acting by the group $\mathcal{H}$ (Remark 3.1) it's immediate that there are

$$d^{2k} + d^{2k-2} + \cdots + d^2 + 1 = \frac{d^n + d^{n-1} + \cdots + d + 1}{d+1}$$

invariant projective lines, where $n = 2k + 1$.

It's worth remarking that the direction of such a projective line at a point $p_i \in sing(\mathcal{F}_0^d)$ is precisely the eigendirection associated to the eigenvalue $\lambda_j^i = (-1 + d\omega^j)\xi^{id}$ with $\omega^j = -1$.

To show that these are not persistent we must bring in the perturbed vector field

$$X_\mu^d = \mu R + X_0^d \qquad \mu \in \mathbf{C}$$

So let's consider some facts about the foliation $\mathcal{F}_\mu^d$, for $0 \leq \mid \mu \mid << 1$. Its singular set $sing(\mathcal{F}_\mu^d)$ consists of $D = d^n + d^{n-1} + \cdots + d + 1$ points $p_{i,\mu} = (x_{1,i,\mu}, \ldots, x_{n,i,\mu})$ and clearly these depend holomorphically on μ for $\mid \mu \mid$ sufficiently small. The coordinates of $p_{i,\mu}$ are given by

$$x_{n-\ell,i,\mu} = (x_{1,i,\mu}^d - \mu)^{-(d^\ell + \cdots + 1)} \qquad 0 \leq \ell \leq n-1$$

in particular

$$(*)_1 \qquad x_{1,i,\mu}(x_{1,i,\mu}^d - \mu)^{D-d^n} = 1$$

Differentiating $(*)_1$ we get

$(*)_2$

$$\frac{\mathbf{d}}{\mathbf{d}\mu}(x_{1,i,\mu})_{|\mu=0} = \left(\frac{D-d^n}{D}\right) x_{1,i,0}^{-(d-1)} = \left(\frac{D-d^n}{D}\right) \xi^{-i(d-1)} \qquad 1 \leq i \leq D$$

where $\xi^D = 1$, ξ primitive.

Let us now return to the invariant projective lines for $\mathcal{F}_0^d$. Again, by considering the group $\mathcal{H}$, it's enough to show that the line

$$L(t) = (1 + t(\varrho - 1), 1, 1 + t(\varrho - 1), \ldots, 1, 1 + t(\varrho - 1))$$

where ϱ is a primitive root of unity of order $d+1$ and n is odd is not persistent. This invariant line contains the points

$$q_{i,0} = (\varrho^i, 1, \varrho^i, \ldots, 1, \varrho^i) \qquad 0 \leq i \leq d$$

Let $q_{i,\mu} \in sing(\mathcal{F}_\mu^d)$ denote the points arising from the $q_{i,0}$. If the projective line persisted then we would have

$$t_2(q_{1,\mu} - q_{0,\mu}) = q_{2,\mu} - q_{0,\mu}$$

which gives t_2 as a holomorphic function of μ. Now, by considering the first and the last coordinates of the above equation, eliminating t_2, differentiating with respect to μ and evaluating at $\mu = 0$ (using $(*)_2$) we get

$$(\varrho^4 - 1)(\varrho - 1) + (D - d^n)(\varrho^2 - 1)^2 = (D - d^n)(\varrho^4 - 1)(\varrho - 1) + (\varrho^2 - 1)^2$$

and this holds if and only if either $n = 1$ or $d = 1$.

Hence, for $n > 1$, $d \geq 2$ and $0 <| \mu |<< 1$, $sing(\mathcal{F}_\mu^d)$ does not have $d+1$ points aligned.

Proposition 3.5. *If $n \geq 2$, $d \geq 2$ and $0 <| \mu |<< 1$, then $\mathcal{F}_\mu^d$ has no algebraic solution.*

Proof. First note that since the eigenvalues of $\mathcal{F}_\mu^d$ at a singular point depend holomorphically on μ then, for $| \mu |$ sufficiently small, all singularities of $\mathcal{F}_\mu^d$ satisfy the hypothesis of Proposition 2.5.

So assume that Γ_μ is an irreducible algebraic curve whose singularities, in case they exist, are such that Γ_μ has only smooth analytic branches, no two of which are tangent, through each of them. Suppose Γ_μ is invariant by $\mathcal{F}_\mu^d$. Let us run through the proof of Proposition 3.3 again. By (1) and (3) of Proposition 3.3 we have

$$\begin{aligned}\sum_{q_{i,\mu}} Res_{\mathcal{F}_\mu^d}(c_1, \Gamma_\mu, q_{i,\mu}) &= \sum_{q_{i,\mu}} \left(-r(i) + \sum_{m=1}^{r(i)} \frac{n+d}{1-d\omega^{j_m}} \right) + \Theta(\mu) \\ &= (n+d)d^0(\Gamma_\mu) - \sum_{q_{i,\mu}} r(i)\end{aligned}$$

where Θ is a holomorphic function of μ. This gives

$$d^0(\Gamma_\mu) = \frac{1}{n+d}\left(\Theta(\mu) + \sum_{q_{i,\mu}} \sum_{m=1}^{r(i)} \frac{n+d}{1-d\omega^{j_m}} \right)$$

There are two possibilities, namely

(i) Θ is not constant as a function of μ.

(ii) Θ is constant as a function of μ (in this case $\Theta \equiv 0$).

In the first possibility, $d^0(\Gamma_\mu)$ cannot be a positive integer for μ close to 0 and $\mu \neq 0$, a contradiction. In the second possibility we repeat the arguments in the proof of Proposition 3.3 and the result follows for $g > 0$. Now, if $g = 0$, then (9) and (10) imply that there are $d+1$ points aligned in $sing(\mathcal{F}_\mu^d)$, and this contradicts Lemma 3.4.

This finishes the proof of Theorem I.

4. Proof of Theorem II

Let $\aleph_d$ denote the space of one-dimensional foliations on $\mathbf{C}P(n)$ of degree $d \geq 2$ and let $\Xi_d \subset \aleph_d$ be the set of non-degenerated foliations of degree d , i.e. , foliations with non-zero eigenvalues at each singularity.

Remark 4.1. Note that by Theorem 2.1 such a foliation has precisely $D = d^n + d^{n-1} + \cdots + d + 1$ singularities. In fact , by taking $\alpha_1 = \cdots =$

$\alpha_{n-1} = 0$, $\alpha_n = 1$ and $\mathcal{L}$ the hyperplane bundle in $\mathbf{C}P(n)$ we have

$$\int_{\mathbf{C}P(n)} c_n(\mathbf{C}P(n) - (\mathcal{L}^{-1})^{\otimes(d-1)}) = \sum_{j=0}^{n} \binom{n+1}{n-j}(d-1)^j =$$

$$= d^n + d^{n-1} + \cdots + d + 1 = \sum \frac{det\mathcal{J}_p}{det\mathcal{J}_p}$$

where the summation extends over all singularities of the foliation.

The proof of Theorem II runs as follows. First we show that Ξ_d is open, dense and connected in $\aleph_d$. Then we show that $\Xi_d{}^*$, the subset of Ξ_d consisting of those foliations whose singular set does not have $d+1$ points aligned and whose linear part at each singularity has distinct eigenvalues, is also open, dense and connected in $\aleph_d$. Next we prove that the eigenvalues of the linear part of a foliation in $\Xi_d{}^*$, at a singular point, can be defined locally as holomorphic functions of the foliation. Then we consider the condition on the eigenvalues, namely $\frac{\lambda_i}{\lambda_j} \notin \mathbf{R}^+$, and show that the subset of $\Xi_d{}^*$ consisting of the foliations which satisfy it, is open, dense and connected. Finally we treat the conditions on the residues and, by using the arguments in the proof of Proposition 3.3, we define $\Im_d$ in a neighborhood of $\mathcal{F}^d_\mu$. It's worth to point out that, in order to define $\Im_d$, it's enough to say what is $\Im_d$ in a neighborhood of $\mathcal{F}^d_\mu$, for $n \geq 2$ and $d \geq 2$, and then use a simple argument of analytic continuation.

The next lemma is a straightforward generalization of Lemma 5 [10], so we omit the proof.

Lemma 4.2. *Ξ_d is open, dense and connected in $\aleph_d$. Moreover, given $\mathcal{F}_0 \in \Xi_d$ with $sing(\mathcal{F}_0) = \{p_1, \ldots, p_D\}$ there are neighborhoods $\mathcal{U}_0$ of $\mathcal{F}_0$ in $\aleph_d$, V_j of p_j in $\mathbf{C}P(n)$ and analytic functions $\psi_j : \mathcal{U}_0 \longrightarrow V_j$, $j = 1, \ldots, D$ such that $V_i \cap V_j = \emptyset$, $i \neq j$, and for any $\mathcal{F} \in \mathcal{U}_0$, $\psi_j(\mathcal{F})$ is the unique singularity of $\mathcal{F}$ in V_j.*

Given $\mathcal{F}_0 \in \Xi_d$ let $\mathcal{U}_0$ and $\psi_j : \mathcal{U}_0 \longrightarrow V_j$ be as in Lemma 4.2 and consider the maps

$$\Psi_j : \mathcal{U}_0 \longrightarrow \mathbf{C}^n \qquad j = 1, \ldots, D$$

defined by

$$\Psi_j(\mathcal{F}) = (tr\partial_2 F(\mathcal{F}, \psi_j(\mathcal{F})), \ldots, tr\wedge^n \partial_2 F(\mathcal{F}, \psi_j(\mathcal{F})))$$

The components of Ψ_j are the elementary symmetric functions of the eigenvalues of the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$. If we let $\triangle$ denote the discriminant variety of monic polynomials of degree n, then the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$ has a repeated eigenvalue if and only if $\Psi_j(\mathcal{F}) \in \triangle$. Since, by Theorem I, there exists $\mathcal{F} \in \Xi_d$ whose linear part at each singularity has distinct eigenvalues and Ξ_d is open and connected, we have that ${\Psi_j}^{-1}(\Delta)$ is an analytic subset of $\mathcal{U}_0$ of codimension ≥ 1, $j = 1, \ldots, D$. Also, it's immediate that the foliations whose singular set has $d+1$ points aligned form an analytic subset of $\mathcal{U}_0$ of codimension ≥ 1. Hence, if ${\Xi_d}^* \subset \Xi_d$ is the subset consisting of foliations not having $d+1$ points aligned and whose linear part at each singularity has distinct eigenvalues, then ${\Xi_d}^* \subset \aleph_d$ is open, dense and connected.

Let

$$\gamma : \mathbf{C}^n \longrightarrow \mathbf{C}^n$$
$$(\lambda_1, \ldots, \lambda_n) \longmapsto (\sigma_1, \ldots, \sigma_n)$$

where σ_i, $i = 1, \ldots, n$ are the elementary symmetric functions of $\lambda_1, \ldots, \lambda_n$ and

$$\mathcal{D} = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n : \exists \ 1 \leq i, j \leq n, \ \ i \neq j, \ \ \lambda_i = \lambda_j\}$$

Then

$$\gamma_{|\mathbf{C}^n \setminus \mathcal{D}} : \mathbf{C}^n \setminus \mathcal{D} \longrightarrow \mathbf{C}^n \setminus \Delta$$

is locally biholomorphic. Given $\mathcal{F}_0 \in {\Xi_d}^*$ choose a neighborhood $W_j \subset \mathbf{C}^n$ of $\Psi_j(\mathcal{F}_0)$ in which a local inverse δ_j of γ is defined and let ${\mathcal{U}_0}^* \subset \mathcal{U}_0 \cap {\Xi_d}^*$ be an open and connected set such that $\Psi_j({\mathcal{U}_0}^*) \subset W_j$, for $j = 1, \ldots, D$.

Define

$$\Phi_j : {\mathcal{U}_0}^* \longrightarrow \mathbf{C}^n \qquad \Phi_j = \delta_j \circ \Psi_{j|{\mathcal{U}_0}^*}$$

Then

$$\Phi_j(\mathcal{F}) = (\lambda_1(\psi_j(\mathcal{F})) \ , \ldots, \ \lambda_n(\psi_j(\mathcal{F})))$$

where $\lambda_i(\psi_j(\mathcal{F}))$, $i = 1, \ldots, n$ are the eigenvalues of the linear part of $\mathcal{F}$ at $\psi_j(\mathcal{F})$.

First we consider the condition $\frac{\lambda_i}{\lambda_j} \notin \mathbf{R}^+$.

Let

$$\rho_{i,k} : \mathbf{C}^n \longrightarrow \mathbf{C} \qquad 1 \leq i \neq k \leq n$$

be defined by

$$\rho_{i,k}(\lambda_1, \ldots, \lambda_n) = \frac{\lambda_k}{\lambda_i}$$

and consider the composites

$$\rho_{i,k} \circ \Phi_j : {\mathcal{U}_0}^* \longrightarrow \mathbf{C} \qquad 1 \leq i \neq k \leq n \ , \ 1 \leq j \leq D$$

Then

$$\mathcal{S} = \{\mathcal{F} \in {\mathcal{U}_0}^* : \ \ Re(\rho_{i,k} \circ \Phi_j(\mathcal{F})) > 0 \ , \ Im(\rho_{i,k} \circ \Phi_j(\mathcal{F})) = 0, \ \ \forall i, j, k\}$$

is such that

$$\mathcal{W} = {\mathcal{U}_0}^* \setminus \mathcal{S}$$

is open, dense and connected in $\mathcal{U}_0{}^*$. Observe that if $\mathcal{F} \in \mathcal{W}$ and $\mathcal{F}$ admits an algebraic solution then such a curve has only smooth analytic branches through each of its singularities, by Proposition 2.5.

Let us treat now the condition on the residues.

Put

$$\Lambda_j^i(\mathcal{F}) = \frac{\sum_{k=1}^n \lambda_k(\psi_j(\mathcal{F}))}{\lambda_i(\psi_j(\mathcal{F}))}$$

Then

$$\Lambda_j^i(\mathcal{F}) = Res_{\mathcal{F}}(c_1, i, \psi_j(\mathcal{F})) + 1$$

where $Res_{\mathcal{F}}(c_1, i, \psi_j(\mathcal{F}))$ denotes the residue of $\mathcal{F}$ at $\psi_j(\mathcal{F})$ in the direction associated to the eigenvalue $\lambda_i(\psi_j(\mathcal{F}))$. Let S_D be a subset of $\{1, \ldots, D\}$ and for $j \in S_D$ let $1 \leq r(j) \leq n$. Then

$$\sum_{i=1}^{r(j)} \Lambda_j^i(\mathcal{F}) = r(j) + \sum_{i=1}^{r(j)} Res_{\mathcal{F}}(c_1, i, \psi_j(\mathcal{F}))$$

Now , given $\mathcal{F}$, if there exists a curve $\Gamma \subset \mathbf{C}P(n)$ invariant by $\mathcal{F}$ we would have, by Proposition 2.8,

$$\chi(\Gamma^\star) = \sum_{j \in S_D} r(j) - (d-1)d^0(\Gamma)$$

and by Proposition 2.7

$$\sum_{j \in S_D} \sum_{i=1}^{r(j)} Res_{\mathcal{F}}(c_1, i, \psi_j(\mathcal{F})) = (n+1)d^0(\Gamma) - \chi(\Gamma^\star)$$

Combining these two equalities we get

$$\sum_{j \in S_D} \sum_{i=1}^{r(j)} Res_{\mathcal{F}}(c_1, i, \psi_j(\mathcal{F})) = (n+d)d^0(\Gamma) - \sum_{j \in S_D} r(j)$$

so that

$$\sum_{j \in S_D} \sum_{i=1}^{r(j)} \Lambda_j^i(\mathcal{F}) = (n+d)d^0(\Gamma)$$

where $d^0(\Gamma) = \frac{(\sum_{j \in S_D} r(j)) - 2 + 2g}{d-1}$.

Let $\mathcal{G}_d$ denote either $\mathcal{F}_0^d$ in case n is even or $\mathcal{F}_\mu^d$, where μ is chosen in such a way that $\mathcal{F}_\mu^d$ has no algebraic solution.

It was shown in Theorem I (propositions 3.3 and 3.5) that $\sum_{j\in S_D}\sum_{i=1}^{r(j)}\Lambda_j^i(\mathcal{G}_d)$ is never a positive integer of the form $(n+d)\beta$, where

$$\beta = \frac{(\sum_{j\in S_D} r(j)) - 2 + 2g}{d-1} \in \mathbf{Z}^+, \quad 1 \leq \sum_{j\in S_D} r(j) \leq nD, \quad g > 0$$

and, by (8) (see the proof of Theorem I), this sum is $n+d$ if, and only if, $g=0$, and this gives $\sum_{j\in S_D} r(j) = d+1$ and there are precisely $d+1$ singularities aligned, which is not the case.

Define a configuration $\mathcal{C}$ to be a pair (S_D, r), where $S_D \neq \emptyset$ is a subset of $\{1,\ldots,D\}$ and r is a function $r: S_D \longrightarrow \{1,\ldots,n\}$. To each configuration $\mathcal{C}$ associate the set $\mathcal{Z}(\mathcal{C}) \subset \mathbf{C}$ defined by

$$\mathcal{Z}(\mathcal{C}) = \{(n+d)\frac{(\sum_{j\in S_D} r(j)) - 2 + 2g}{d-1} : \quad g \in \mathbf{Z}^+\}$$

where $\mathcal{C} = (S_D, r)$.

Let $\mathcal{W}$ be the neighborhood of $\mathcal{G}_d$ obtained above. For each configuration $\mathcal{C}$, the holomorphic function

$$\Theta_{\mathcal{C}} \equiv \left(\sum_{j\in S_D}\sum_{i=1}^{r(j)}\Lambda_j^i\right)_{|\mathcal{W}} : \mathcal{W} \longrightarrow \mathbf{C}$$

is such that

$$\Theta_{\mathcal{C}}(\mathcal{G}_d) \in \mathbf{C}\setminus\mathcal{Z}(\mathcal{C})$$

Therefore, the same holds for any $\mathcal{F}$ in a neighborhood $\mathcal{N}(\mathcal{C},\mathcal{G}_d)$ of $\mathcal{G}_d$ in $\mathcal{W}$. Since the number of configurations is finite we let

$$\mathcal{W}_1 = \bigcap\nolimits_{\mathcal{C}}\mathcal{N}(\mathcal{C},\mathcal{G}_d) \subset \mathcal{W}$$

where the intersection is taken over all configurations. It remains to consider the case of invariant projective lines. Given a configuration $\mathcal{C}$, look at

$$(\Theta_{\mathcal{C}})_{|\mathcal{W}_1} : \mathcal{W}_1 \longrightarrow \mathbf{C}$$

It follows from (9) in the proof of Theorem I that

$$Re((\Theta_{\mathcal{C}})_{|\mathcal{W}_1}(\mathcal{G}_d)) \leq n+d$$

If $(\Theta_{\mathcal{C}})_{|\mathcal{W}_1} \equiv n+d$ then (10) in the proof of Theorem I implies that there are $d+1$ singularities aligned, which does not hold in $\Xi_d{}^*$. If this function is not constant, then $(\Theta_{\mathcal{C}})_{|\mathcal{W}_1}$ can never be an integer of the form $(n+d)\beta$ whith $\mathbf{Z}^+ \ni \beta \geq 2$ in a neighborhood $\mathcal{W}_2$ of $\mathcal{G}_d$ and, those

$\mathcal{F}$ in this neighborhood, if any, for which this function equals $n+d$ do not have $d+1$ singularities aligned. We let

$$\Im_d \cap \mathcal{W} = \mathcal{W}_2$$

Now, using the fact that $\Xi_d{}^*$ is open, dense and connected, a simple argument of analytic continuation shows that $\Im_d$ is open and dense. Theorem II is proved.

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